

A SHARP INEQUALITY AND ITS APPLICATIONS

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ABSTRACT. We establish an analog Hardy inequality with sharp constant involving exponential weight function. The special case of this inequality (for $n = 2$) leads to a direct proof of Onofri inequality on S^2 .

1. INTRODUCTION

The classical Hardy inequality says that for any non-negative function $f(x)$ on $[0, +\infty)$, if $F(x) = \int_0^x f(t)dt$, then

$$\int_0^\infty \left(\frac{F}{x}\right)^k dx \leq \left(\frac{k}{k-1}\right)^k \int_0^\infty f^k dx,$$

where $k > 1$ is a given parameter. See, for example, Inequality 327 in the book by Hardy, Littlewood and Polya [9]. It is important to note that the constant $(\frac{k}{k-1})^k$ is the optimal one. Using Hölder inequality Hardy and Littlewood were able to derive that

$$\int_0^\infty \frac{F^l}{x^{l-\alpha}} dx \leq \left(\frac{k}{k-1}\right)^k \left(\int_0^\infty f^k dx\right)^{\frac{l}{k}},$$

where $l \geq k$ and $\alpha = l/k - 1$. It was quite clear to them that the constant is not optimal for $l > k$. Though they guessed what is the best constant, it was later proved by Bliss, who obtained nowadays the famous Bliss Lemma (see the interesting papers [8] and [3]):

Bliss Lemma: *Let k, l be constants, such that $l > k > 1$, and let $f(x)$ be a non-negative measurable function in the intervals $0 \leq x < \infty$, such that the integral $J = \int_0^\infty f^k dx$ is finite. Then the integral $y = \int_0^x f dx$ is finite for every x and*

$$(1.1) \quad I = \int_0^\infty \frac{y^l}{x^{l-\alpha}} dx \leq C_b J^{l/k},$$

where

$$\alpha = \frac{l}{k} - 1, \quad C_b = \frac{1}{l - \alpha - 1} \left[\frac{\alpha \Gamma(l/\alpha)}{\Gamma(1/\alpha) \Gamma((l-1)/\alpha)} \right]^\alpha.$$

The equality in (1.1) holds if and only if $f(x) = c/(1+dx^\alpha)^{(\alpha+1)/\alpha}$ for some positive constants c, d .

Bliss Lemma later (after more than forty years) became a crucial ingredient in the proof of sharp Sobolev inequality by Aubin [1], and Talenti [13] respectively. The latter inequality has played essential role in the resolution of the Yamabe problem, which mainly concerns about finding a canonical metrics with constant scalar curvature on compact manifolds with dimension higher than or equal to three (see the geometric and analytic forms of sharp Sobolev inequalities in the appendix).

The Yamabe problem can also be viewed as the higher dimensional analogue to the uniformization theorem for two dimensional manifolds. The analytic approach to the re-proof of the uniformization theorem seems to be initiated by Berger [2]. While for the Yamabe problem for manifold with positive Yamabe constant, one seeks a new metric in the same conformal class (with fixed volume) which yields the smallest total scalar curvature, in the analytic approach to the re-proof of the uniformization theorem on topological spheres one looks for a new metric in the same conformal class (with fixed area) which has the smallest Liouville energy, see, for example, Hamilton [7], Chow [6], or the recent paper by Chen and Zhu [5]. The core inequality in such an argument is the Onofri inequality (see the precise form in the appendix). Recently we showed in [10] that one can derive the Onofri inequality directly from Trudinger's inequality. Comparing the proof of sharp Sobolev inequality with that of Onofri inequality, we feel that there is an undiscovered calculus inequalities, which turns out to be the main theorem of this paper.

Theorem 1. 1). Let $n > 1$ be given. For any nonnegative function $u \in C^1[0, +\infty)$ with $u(0) = 0$,

$$(1.2) \quad \ln \int_0^{+\infty} \frac{e^{nu}}{e^{nr}} dr \leq \left(\frac{n-1}{n}\right)^{n-1} \int_0^{+\infty} |u_r|^n dr + C_n,$$

where the constant C_n is given by

$$C_n = \int_0^1 \frac{1}{t} (1 - (1-t)^{n-[n]}) dt + \sum_{i=1}^{[n]-1} \frac{1}{(n-i)},$$

$[n]$ is the integer part of n . Both constants $\left(\frac{n-1}{n}\right)^{n-1}$ and C_n are optimal, and the equality never holds.

2). For any nonnegative function $u \in C^1[0, +\infty)$ with $u(0) = 0$,

$$(1.3) \quad \ln \int_0^{+\infty} \frac{e^u}{e^r} dr \leq \int_0^{+\infty} |u_r| dr.$$

We first prove the above inequality with a larger coefficient in Section 2 (Proposition 1 below). The argument is elementary and simple. It needs to be pointed out that for $n > 1$ being an integer, Theorem 1 can be read out from Theorem 1.3 in [10]. For general positive constant, it seems impossible to prove Theorem 1 from that theorem, rather, Theorem 1 provides an alternative proof of that theorem (Corollary 3 in this paper). Recall the original proof of Theorem 1.3 in [10] does rely on Trudinger's inequality. Quite interestingly, we also recall that Moser [11] used a similar argument to give a very simple proof of the improved Trudinger's inequality (with best constant):

Corollary 1. (Weak Moser's inequality) Let $\Omega \subset \mathbf{R}^n$ (for $n \geq 2$) be a smooth bounded domain. For any $\beta < n\omega_{n-1}^{1/(n-1)}$, there is a constant $C(\Omega, \beta)$ depending on the volume of Ω and β , such that for all $u \in W_0^{1,n}(\Omega)$ with $\int_{\Omega} |\nabla u|^n dx \leq 1$,

$$\int_{\Omega} e^{\beta u^{\frac{n}{n-1}}} dx \leq C(\Omega, \beta).$$

Here and throughout this paper, we use ω_n for the volume of unit sphere S^n in R^{n+1} . This result is slightly weaker than Moser's inequality since it does not include the case of $\beta = n\omega_{n-1}^{1/(n-1)}$. It seems that one needs the argument due to Moser [11], or Carleson and Chang [4] to cover this extremal case.

In Section 3 we will show how to improve the rough inequality (Proposition 1) and complete the proof of the main theorem. One particular reason that we can achieve this (but not for Moser's inequality) is that we can classify all extremal functions.

As the Bliss Lemma yields sharp Sobolev inequality, in Section 4 we will show that Theorem 1 can be used to give a more direct proof of the Onofri inequality (thus without even using Trudinger's inequality). In fact, let $B_r(0) \subset R^n$ (now n is an integer greater than or equal to two) be a ball in R^n with radius r centered at the origin, and

$$D_a^b(B_r(0)) = \{f(y) : f(y) - b \in W_0^{1,n}(B_r(0)), \int_{B_r(0)} e^{nf} dy = a\},$$

where a is a constant satisfying $a > \frac{\omega_{n-1} r^n e^{nb}}{n}$. We will show that Theorem 1 yields

Corollary 2. (Local sharp inequality for $n = 2$)

$$\inf_{w \in D_a^b(B_r)} \int_{B_r} |\nabla w|^2 dy = 4\pi \cdot \left(\ln \frac{ae^{-2b}}{\pi r^2} + \frac{\pi r^2}{ae^{-2b}} - 1 \right).$$

It is known now that this corollary implies Onofri inequality on S^2 , see, Li and Zhu [10]. For readers' convenience we include a complete proof of the Onofri inequality in Section 4.

In Section 4 we shall also discuss the applications of the main theorem to other geometric problems. For readers' convenience, we present both geometric and analytic forms of sharp Sobolev inequality on S^n (for $n \geq 3$) and Onofri inequality on S^2 in the appendix.

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2. ROUGH INEQUALITY

We shall establish two elementary calculus inequalities in this section. The first one will be used to prove the main theorem, and the second one will be used to derive Corollary 1.

Proposition 1. (1). Let $n > 1$ and $\beta_0 > \left(\frac{n-1}{n}\right)^{\frac{n-1}{n}}$. There is a constant $c_1(\beta_0)$, such that for any $u(r) \in C^1[0, +\infty)$ satisfying $u(0) = 0$,

$$(2.1) \quad \ln \int_0^\infty e^{n(u-r)} dr \leq \beta_0^n \int_0^\infty |u_r|^n dr + c_1(\beta_0).$$

(2). For $u(r) \in C^1[0, +\infty)$ satisfying $u(0) = 0$,

$$\ln \int_0^\infty \frac{e^u}{e^r} dr \leq \int_0^\infty |u_r| dr.$$

For $n = 1$, the above is an optimal inequality. For $n \geq 2$ we will improve the inequality by variational method in next section.

Proof. Let $u(r)$ be any function in $C^1[0, +\infty)$ satisfying $u(0) = 0$. We have

$$u(r) \leq \int_0^r |u_r| dr,$$

thus

$$\int_0^r \frac{e^u}{e^r} dr \leq \exp\left\{\int_0^r |u_r| dr\right\},$$

which yields

$$\ln \int_0^r \frac{e^u}{e^r} dr \leq \int_0^r |u_r| dr.$$

Now, for given $n > 1$ and positive parameter $\beta > 0$, we have

$$\begin{aligned} u(r) &= \int_0^r u_r dr \leq \left(\int_0^r |u_r|^n dr\right)^{1/n} \cdot r^{\frac{n-1}{n}} \\ &\leq \frac{\beta^n \int_0^r |u_r|^n dr}{n} + \frac{\beta^{-\frac{n}{n-1}} r}{n/(n-1)}. \end{aligned}$$

Thus

$$\begin{aligned} (2.2) \quad \int_0^r \frac{e^{nu}}{e^{nr}} dr &\leq \int_0^r \frac{\exp\{\beta^n \int_0^r |u_r|^n dr + (n-1)\beta^{-\frac{n}{n-1}} r\}}{e^{nr}} dr \\ &= \exp\{\beta^n \int_0^r |u_r|^n dr\} \cdot \int_0^r e^{[(n-1)\beta^{-\frac{n}{n-1}} - n]r} dr. \end{aligned}$$

If we choose

$$(2.3) \quad \beta = \beta_0 > \left(\frac{n-1}{n}\right)^{\frac{n-1}{n}},$$

then

$$\int_0^r e^{[(n-1)\beta_0^{-\frac{n}{n-1}} - n]r} dr = c(\beta_0)$$

is a finite number depending on β_0 . It follows that

$$\ln \int_0^r e^{n(u-r)} dr \leq \beta_0^n \int_0^r |u_r|^n dr + c_1(\beta_0)$$

for $c_1(\beta_0) = \ln c(\beta_0)$. □

It is obvious in the above proof that $c(\beta_0), c_1(\beta_0) \rightarrow +\infty$ as $\beta_0 \rightarrow \left(\frac{n-1}{n}\right)^{\frac{n-1}{n}}$. We need another argument to derive the main theorem.

Remark 1. From (2.2) we can see that for β_0 satisfying (2.3),

$$\begin{aligned} (2.4) \quad \int_R^r \frac{e^{nu}}{e^{nr}} dr &\leq \exp\{\beta_0^n \int_0^r |u_r|^n dr\} \cdot \int_R^r e^{[(n-1)\beta_0^{-\frac{n}{n-1}} - n]r} dr \\ &= o_R(1) \exp\{\beta_0^n \int_0^r |u_r|^n dr\}, \end{aligned}$$

where $o_R(1) \rightarrow 0$ as $R \rightarrow \infty$.

We now compare this with Moser's proof of Trudinger's inequality

Lemma 1. For $n > 1$, $a > 0$ and $\beta < na^{\frac{1}{1-n}}$, there is a constant $C_{\beta,a}$ depending only on β and a , such that for any nonnegative function $u \in C^1[0, +\infty)$ with $u(0) = 0$ and $\int_0^\infty |u_r|^n dr \leq a$,

$$\int_0^\infty \frac{e^{\beta u^{\frac{n}{n-1}}}}{e^{nr}} dr \leq C_{\beta,a}.$$

Proof. For given $n > 1$ we have

$$u(r) = \int_0^r u_r dr \leq \left(\int_0^r |u_r|^n dr \right)^{1/n} \cdot r^{\frac{n-1}{n}} \leq a^{\frac{1}{n}} r^{\frac{n-1}{n}}.$$

Thus for any positive parameter $\tau > 0$,

$$(2.5) \quad \int_0^\infty \frac{e^{\tau u^{\frac{n}{n-1}}}}{e^{nr}} dr \leq \int_0^\infty \exp\{\tau a^{\frac{1}{n-1}} - n\} r dr.$$

The right hand side of the above inequality is bounded if we choose $\tau = \beta < na^{\frac{1}{1-n}}$. \square

Based on Lemma 1, one can verify Corollary 1 as follows.

Due to the rearrangement and rescaling, we only need to prove Corollary 1 when $\Omega = B_1(0)$ and $u \in C_0^1(B_1(0))$ is radially symmetric and nonnegative.

From $\int_{B_1} |\nabla u|^n dx \leq 1$, we know that (let $r = -\ln s$)

$$1 \geq \int_{B_1} |\nabla u|^n dx = \omega_{n-1} \int_0^1 |u_s|^n s^{n-1} ds = \omega_{n-1} \int_0^\infty |u_r|^n dr.$$

Also,

$$\int_{B_1} e^{\beta u^{\frac{n}{n-1}}} dx = \omega_{n-1} \int_0^1 e^{\beta u^{\frac{n}{n-1}}} s^{n-1} ds = \omega_{n-1} \int_0^\infty \frac{e^{\beta u^{\frac{n}{n-1}}}}{e^{nr}} dr.$$

One immediately has Corollary 1 by using Lemma 1 with $a = \omega_{n-1}^{-1}$.

3. SHARP INEQUALITY

We shall prove the main theorem in this section. Since the case of $n = 1$ has been settled by Proposition 1, we will focus on the case of $n > 1$. For given $a > 0$, define

$$(3.1) \quad D_a^n := \{u(r) \in W^{1,n}(R^+) : u(0) = 0, \int_0^\infty \exp\{nu - nr\} dr = a\}.$$

Lemma 2. There is a $v \in D_a^n$ such that

$$\int_0^\infty |v_r|^n dr = \inf_{u \in D_a^n} \int_0^\infty |u_r|^n dr := \xi.$$

Proof. Let $\{v^i\}$ be a minimizing sequence of $\inf_{u \in D_a^n} \int_0^\infty |u_r|^n dr$. Then

$$v^i \rightharpoonup v \quad \text{in } W^{1,n}(R^+), \quad \text{and} \quad \int_0^\infty |v_r|^n dr \leq \liminf_{i \rightarrow \infty} \int_0^\infty |v_r^i|^n dr = \xi$$

for some $v \in W^{1,n}(R^+)$. We need to verify $v \in D_a^n$.

First, from (2.4), we know that for $w = v^i$, or v :

$$\int_R^\infty \frac{e^{nw}}{e^{nr}} dr = o_R(1).$$

On the other hand, it follows from the embedding $H^1(0, R) \hookrightarrow C^{0,1/2}(0, R)$ and Arzela-Ascoli lemma that

$$\lim_{i \rightarrow \infty} \int_0^R \exp\{nv^i - nr\} dr = \int_0^R \exp\{nv - nr\} dr.$$

Letting $i, R \rightarrow \infty$, we have $\int_0^\infty \exp\{nv - nr\} dr = a$, that is $v \in D_a^n$. \square

We now begin the proof of the main theorem.

Proof. We only need to consider nontrivial nonnegative functions. For $a > 1/n$, let v be the minimizer of $\inf_{u \in D_a^n} \int_0^\infty |u_r|^n dr$. It is easy to see that $v_r \geq 0$. So it satisfies the following Euler-Lagrange equation:

$$(3.2) \quad v_r^{n-2} v_{rr} = -\tau e^{nv-nr}, \quad v(0) = 0$$

for some $\tau > 0$. Though it is not obvious how to obtain the general solution from the uniqueness of the ordinary differential equation since v_r could be zero, one can follow the argument given by Carleson and Chang ([4], page 123) to show that the general solution to (3.2) is given by

$$v(r) = \ln \frac{1}{\lambda_0 + e^{-nr/(n-1)}} - \frac{1}{n} \ln \frac{\tau}{(\frac{n}{n-1})^n \lambda_0},$$

where λ_0 is a positive constant and $\tau = \frac{(\frac{n}{n-1})^n \lambda_0}{(\lambda_0 + 1)^n}$. Thus

$$(3.3) \quad v(r) = \ln \frac{\lambda_0 + 1}{\lambda_0 + e^{-nr/(n-1)}}.$$

Since $a = \int_0^\infty e^{nv-nr} dr$, we have

$$\begin{aligned} a &= \int_0^\infty \left(\frac{\lambda_0 + 1}{\lambda_0 + e^{-nr/(n-1)}} \right)^n e^{-nr} dr \\ &= \int_0^1 \left(\frac{\lambda_0 + 1}{\lambda_0 + s^{n/(n-1)}} \right)^n s^n \left(\frac{1}{s} \right) ds \\ &= (\lambda_0 + 1)^n \int_0^1 \frac{s^{n-1}}{(\lambda_0 + s^{n/(n-1)})^n} ds \\ &= (\lambda_0 + 1)^n \frac{s^n}{n\lambda_0(\lambda_0 + s^{n/(n-1)})^{n-1}} \Big|_{s=0}^1 \\ &= \frac{\lambda_0 + 1}{n\lambda_0}, \end{aligned}$$

That is

$$(3.4) \quad \lambda_0 = \frac{1}{na - 1}.$$

We compute

$$\begin{aligned}
\int_0^\infty |v_r|^n dr &= \int_0^\infty \left| \frac{\frac{n}{n-1} e^{-nr/(n-1)}}{\lambda_0 + e^{-nr/(n-1)}} \right|^n dr \\
&= \left(\frac{n}{n-1} \right)^n \int_0^\infty \left(\frac{e^{-nr/(n-1)}}{\lambda_0 + e^{-nr/(n-1)}} \right)^n dr \\
&= \left(\frac{n}{n-1} \right)^{n-1} \int_0^{1/\lambda_0} \frac{t^{n-1}}{(1+t)^n} dt \\
&= \left(\frac{n}{n-1} \right)^{n-1} \int_1^{1/\lambda_0+1} \frac{(\tau-1)^{n-1}}{\tau^n} d\tau \\
&= -\left(\frac{n}{n-1} \right)^{n-1} \int_1^{1/\lambda_0+1} (\tau-1) \left(1 - \frac{1}{\tau}\right)^{n-2} d\frac{1}{\tau} \\
&= \left(\frac{n}{n-1} \right)^{n-1} \int_{\lambda_0/(\lambda_0+1)}^1 \left(\frac{1}{t} - 1\right) (1-t)^{n-2} dt.
\end{aligned}$$

Using (3.4) we have: If $n \in \mathbf{N}$,

$$\begin{aligned}
\int_0^\infty |v_r|^n dr &= \left(\frac{n}{n-1} \right)^{n-1} \cdot \left\{ - \int_{\lambda_0/(\lambda_0+1)}^1 (1-t)^{n-2} dt \right. \\
&\quad \left. - \dots - \int_{\lambda_0/(\lambda_0+1)}^1 (1-t) dt - \int_{\lambda_0/(\lambda_0+1)}^1 \left(1 - \frac{1}{t}\right) dt \right\} \\
&= \left(\frac{n}{n-1} \right)^{n-1} \left\{ \ln \frac{\lambda_0+1}{\lambda_0} - \frac{1}{\lambda_0+1} - \sum_{i=1}^{n-2} \frac{\left(\frac{1}{\lambda_0+1}\right)^{n-i}}{n-i} \right\} \\
&= \left(\frac{n}{n-1} \right)^{n-1} \left\{ \ln(na) - \sum_{i=1}^{n-1} \frac{(na-1)^{n-i}}{(n-i)(na)^{n-i}} \right\};
\end{aligned}$$

For general $n > 1$, we have

$$\begin{aligned}
\int_0^\infty |v_r|^n dr &= \left(\frac{n}{n-1} \right)^{n-1} \cdot \left\{ - \int_{\lambda_0/(\lambda_0+1)}^1 (1-t)^{n-2} dt \right. \\
&\quad \left. - \dots - \int_{\lambda_0/(\lambda_0+1)}^1 (1-t)^{n-[n]} dt + \int_{\lambda_0/(\lambda_0+1)}^1 \frac{1}{t} (1-t)^{n-[n]} dt \right\} \\
&= \left(\frac{n}{n-1} \right)^{n-1} \left\{ \int_{\lambda_0/(\lambda_0+1)}^1 \frac{1}{t} (1-t)^{n-[n]} dt - \sum_{i=1}^{[n]-1} \frac{\left(\frac{1}{\lambda_0+1}\right)^{n-i}}{n-i} \right\} \\
&= \left(\frac{n}{n-1} \right)^{n-1} \left\{ \int_{\lambda_0/(\lambda_0+1)}^1 \frac{1}{t} + \int_{\lambda_0/(\lambda_0+1)}^1 \frac{1}{t} ((1-t)^{n-[n]} - 1) dt - \sum_{i=1}^{[n]-1} \frac{\left(\frac{1}{\lambda_0+1}\right)^{n-i}}{n-i} \right\} \\
&\geq \left(\frac{n}{n-1} \right)^{n-1} \left\{ \ln(na) - \int_{\lambda_0/(\lambda_0+1)}^1 \frac{1}{t} (1 - (1-t)^{n-[n]}) dt - \sum_{i=1}^{[n]-1} \frac{(na-1)^{n-i}}{(n-i)(na)^{n-i}} \right\},
\end{aligned}$$

where $[n]$ is the integer part of n . Let $a \rightarrow \infty$, then $\lambda_0 \rightarrow 0$ by (3.4). We know that C_n is optimal. The proof is completed. \square

Remark 2. For negative function u , we can certainly improve the inequalities. In particular, similar argument will yield Theorem 1.3 (ii) in [10] for integer $n > 1$.

Since we do not have meaningful applications for this inequality so far, we shall skip details here.

4. APPLICATIONS

We shall show in this section that Theorem 1 implies Corollary 2, as well as Theorem 1.3 in [10], and then shall derive the Onofri inequality from Corollary 2.

We first prove Corollary 2. Let $v \in D_\alpha^2$ (recalling the notation in (3.1)). We have, from the proof of Theorem 1, that

$$(4.1) \quad \inf_{v \in D_\alpha^2} \int_0^\infty |v_r|^2 dr \geq 2\{\ln(2\alpha) + \frac{1}{2\alpha} - 1\},$$

where $\int_0^\infty e^{2v-2r} dr = \alpha$. For $w \in D_a^0(B_1(0))$,

$$\int_{B_1} |\nabla w|^2 dx = 2\pi \int_0^1 |w_s|^n s ds = 2\pi \int_0^\infty |w_r|^2 dr.$$

and

$$\int_{B_1} e^{2w} dx = 2\pi \int_0^1 e^{2w} s ds = 2\pi \int_0^\infty \frac{e^{2w}}{e^{2r}} dr.$$

Combing with (4.1), we have

$$\inf_{w \in D_a^0(B_r)} \int_{B_1} |\nabla w|^2 dx = 4\pi \cdot \left(\ln \frac{\int_{B_1} e^{2w} dx}{\pi} + \frac{\pi}{\int_{B_1} e^{2w} dx} - 1 \right).$$

After rescaling and shifting, we get Corollary 2.

In the same spirit, we easily obtain

Corollary 3. *Let $u \in C^1(B_1)$ be a nonnegative function satisfying $u = 0$ on ∂B_1*

$$\ln \frac{n \int_{B_1} e^{nu}}{\omega_{n-1}} < \left(\frac{n-1}{n} \right)^{n-1} \omega_n^{-1} \int_{B_1} |\nabla u|^n + F(1).$$

where

$$F(1) = 1 + \frac{1}{2} + \cdots + \frac{1}{n-1}.$$

The fact that the strict inequality holds on a bounded domain coincides with the one that the strict sharp Sobolev inequality holds on a bounded domain. Corollary 3 was first proved in [10] using Trudinger's inequality. The proof presented here does not rely on Trudinger's inequality. Inequality in Corollary 3 was refereed as *local sharp inequality* in [10], which is easily adapted for manifolds. See related topics in Chen and Zhu [5].

Finally, we shall show that one can prove the Onofri inequality (see both forms of the inequality in appendix) using Corollary 2.

Due to the rearrangement, we only need to prove Onofri inequality for $u \in C^1(S^2)$ which depends only on x_3 and is monotonically decreasing in x_3 . Also, we can assume that $u(x_3)|_{x_3=1} = 0$ (otherwise, we replace $u(x)$ by $u(x) - u(1)$). We can approximate u by a sequence of functions $u_i \in C^1(S^2)$ such that $u_i(x) = u_i(x_3)$ is monotonically decreasing in x_3 , and $u_i(x) = 0$ in the geodesic ball $B_{1/i}(N)$ of the north pole N for $i \in \mathbf{N}$. Denote $S_i^2 := S^2 \setminus B_{1/i}(N)$.

Let $\Phi: x \in S^2 \rightarrow y \in R^2$ be a stereographic projection given by

$$x_i = \frac{2y_i}{1+|y|^2}, \quad \text{for } i = 1, 2;$$

and

$$x_3 = \frac{|y|^2 - 1}{|y|^2 + 1}.$$

Denote

$$g_0 = \sum_{i=1}^3 dx_i^2 = \left(\frac{2}{1+|y|^2}\right)^2 dy^2 := e^{2\varphi(y)} dy^2.$$

Thus

$$\varphi(y) = \ln \frac{2}{1+|y|^2}.$$

It is easy to check that $\varphi(y)$ satisfies

$$(4.2) \quad -\Delta\varphi = e^{2\varphi} \quad \text{in } R^2.$$

Let $\Phi(S_i^2) = B_{R_i}$. It is obvious that $R_i \rightarrow +\infty$ as $i \rightarrow +\infty$. For

$$w_i(y) = u_i(x) + \varphi(y) = u_i(\Phi^{-1}(y)) + \varphi(y),$$

we have

$$\int_{B_{R_i}} e^{2w_i(y)} dy = \int_{S_i^2} e^{2u_i} dx := a_i,$$

and

$$\begin{aligned} \int_{B_{R_i}} |\nabla w_i|^2 dy &= \int_{B_{R_i}} |\nabla(u_i \circ \Phi^{-1})|^2 dy + 2 \int_{B_{R_i}} \nabla(u_i \circ \Phi^{-1}) \cdot \nabla\varphi dy + \int_{B_{R_i}} |\nabla\varphi|^2 dy \\ &= \int_{S_i^2} |\nabla u_i|^2 dx + 2 \int_{S_i^2} u_i dx + \int_{B_{R_i}} |\nabla\varphi|^2 dy, \end{aligned}$$

where we use the fact that φ satisfies (4.2). Since $w_i(y) = \ln \frac{2}{1+R_i^2}$ on ∂B_{R_i} , it follows from Corollary 2 that

$$\int_{B_{R_i}} |\nabla w_i|^2 dy \geq 4\pi \left(\ln \frac{a_i \cdot \left(\frac{1+R_i^2}{2}\right)^2}{\pi R_i^2} + \frac{\pi R_i^2}{a_i \cdot \left(\frac{1+R_i^2}{2}\right)^2} - 1 \right).$$

Also, one can check that

$$\int_{B_{R_i}} |\nabla\varphi|^2 dy = 4\pi \left[\ln(1+R_i^2) + \frac{1}{1+R_i^2} - 1 \right].$$

We conclude

$$\begin{aligned} \int_{S_i^2} |\nabla u_i|^2 dx + 2 \int_{S_i^2} u_i dx &\geq 4\pi \left(\ln \frac{a_i \cdot \left(\frac{1+R_i^2}{2}\right)^2}{\pi R_i^2} + \frac{\pi R_i^2}{a_i \cdot \left(\frac{1+R_i^2}{2}\right)^2} - 1 \right) \\ &\quad - 4\pi \left[\ln(1+R_i^2) + \frac{1}{1+R_i^2} - 1 \right] \\ &= 4\pi \left(\ln \frac{a_i \cdot (1+R_i^2)}{4\pi R_i^2} + \frac{4\pi R_i^2}{a_i \cdot (1+R_i^2)^2} - \frac{1}{1+R_i^2} \right). \end{aligned}$$

Sending $i \rightarrow +\infty$, we have

$$\int_{S^2} |\nabla u|^2 dx + 2 \int_{S^2} u dx \geq 4\pi \left(\ln \frac{1}{4\pi} \int_{S^2} e^{2u} dx \right).$$

5. APPENDIX

For readers' convenience, we include geometric and analytic forms of sharp Sobolev inequality on S^n (for $n \geq 3$), as well as geometric and analytic forms of Onofri inequality on S^2 . These are well-known to experts in the field.

Sharp Sobolev inequality on S^n (for $n \geq 3$): Let (S^n, g_0) be the standard unit sphere in R^{n+1} ($n \geq 3$). For any $u \in H^1(S^n)$,

$$\left(\frac{1}{\omega_n} \int_{S^n} |u|^{\frac{2n}{n-2}} dv_{g_0}\right)^{\frac{n-2}{n}} \leq \frac{1}{\omega_n} \int_{S^n} \left(u^2 + \frac{4}{n(n-2)} |\nabla u|^2\right) dv_{g_0}.$$

The equality holds if and only if the scalar curvature of $u^{\frac{4}{n-2}} g_0$ is constant.

If $\tilde{g} = \rho g$ is a conformal metric to the background metric g , then the new scalar curvature \tilde{R} under metric \tilde{g} satisfies

$$(5.1) \quad \tilde{R} = \rho^{-1} R - (n-1)\rho^{-2} \Delta \rho - \frac{1}{4}(n-1)(n-6)\rho^{-3} |\nabla \rho|^2,$$

where R is the scalar curvature under metric g . If we write $\rho = e^{2u}$, we have

$$(5.2) \quad \tilde{R} = e^{-2u} [R - (n-1)(n-2) |\nabla u|^2 - 2(n-1) \Delta u].$$

The normalized total scalar curvature under metric \tilde{g} is defined by

$$(5.3) \quad E(\tilde{g}) = \frac{\int_{S^n} \tilde{R} dV_{\tilde{g}}}{\left(\int_{S^n} dV_{\tilde{g}}\right)^{(n-2)/n}}.$$

Geometric form of Sharp Sobolev inequality on S^n (for $n \geq 3$): Let (S^n, g_0) be the standard unit sphere in R^{n+1} ($n \geq 3$). Then

$$\inf_{\tilde{g}=\rho g_0} E(\tilde{g}) = n(n-1)\omega_n^{2/n},$$

and the infimum is achieved if and only if \tilde{R} (under metric $\tilde{g} = \rho g_0$) is a constant.

For dimension $n = 2$, we have

Onofri inequality on S^2 : Let (S^2, g_0) be the standard unit sphere in R^3 . For any $u \in W^{1,2}(S^2)$,

$$\ln\left(\frac{1}{4\pi} \int_{S^2} e^{2u} dx\right) \leq \frac{1}{4\pi} \int_{S^2} (|\nabla u|^2 + 2u) dx.$$

The equality holds if and only if the curvature under metric $e^{2u} g_0$ is constant.

Let (M, g) be a smooth Riemann surface. For any conformal new metric $g_1 = e^{2u} g$, the corresponding Liouville energy is defined by

$$L_g(g_1) = \frac{1}{4} \int_M \ln \frac{g_1}{g} \cdot (R_{g_1} dV_{g_1} + R_g dV_g)$$

where R_g and R_{g_1} are twice the Gaussian curvatures K_g and K_{g_1} with respect to metrics g and g_1 . Due to (5.2), the Liouville energy of metric g_1 can also be represented by

$$L_g(g_1) = \int_M (|\nabla_g u|^2 + R_g u) dV_g.$$

Geometric form of Onofri inequality on S^2 : Let (S^2, g_0) be the standard unit sphere in R^3 , and $[g_0]_1 = \{g = \rho g_0, \text{ for some } \rho > 0, \text{ and } \int_{S^2} dV_g = 4\pi\}$. Then

$$\inf_{g \in [g_0]_1} L_{g_0}(g) = 0,$$

and the infimum is achieved if and only if $R = 2$ (under metric $g = \rho g_0$).

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